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ON AN APPROXIMATE METHOD OF SOLUTION OF THE EQUATIONS OF THE THEORY OF IDEAL PLASTICITY

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The axisymmetric problem for an ideally plastic body is considered under the conditions of incomplete plasticity according to the Haar-Kármán hypothesis (the intermediate of the principal stresses, the annular, is related to the strains by Hooke's law, and the strain and stress tensors are coaxial). The corresponding system of equations is quasi-linear and of hyperbolic type. Its characteristics are surfaces of maximal shear stress. An iteration process is proposed for the approximate solution, and its convergence is proved.

1. Equilibrium equations for axial symmetry. Let us examine a body of revolution in a cylindrical r, φ, z coordinate system. We assume the load to be independent of the angle φ. Then, the equilibrium equations are

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\varphi}{r}=0 \qquad (1.1)$$
$$\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=0$$

The principal stresses are

$$\sigma_{1,2}=\frac{\sigma_r+\sigma_z}{2}\pm\frac{1}{2}\sqrt{(\sigma_r-\sigma_z)^2+4\tau_{rz}^2}, \quad \sigma_3=\sigma_\varphi$$

Following [1], we distinguish two possibilities: (1) the circumferential stress σ_φ is extreme ($\sigma_1>\sigma_2>\sigma_\varphi$ or $\sigma_2>\sigma_1>\sigma_\varphi$), (2) σ_φ is the intermediate stress ($\sigma_1>\sigma_\varphi>\sigma_2$ or $\sigma_2>\sigma_\varphi<\sigma_1$).

Henceforth, we shall consider the incomplete state of plasticity for which σ_φ is the intermediate stress. For definiteness, let us assume that

$$\sigma_1>\sigma_\varphi>\sigma_2, \qquad \sigma_1-\sigma_2=1 \qquad (1.2)$$

everywhere in the domain under consideration. All the stresses here are referred to twice the torsion yield point.

Let u, v denote the displacement vector components along the r, z axes, respectively, and let us take the Haar-Kármán hypothesis [1, 2], according to which the stress σ_φ retains the elastic relation with the strains

$$\sigma_\varphi=\frac{3k}{1+\nu}\left[\nu\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial z}\right)+(1-\nu)\frac{u}{r}\right] \qquad (1.3)$$

Here ν is the Poisson's ratio, k the modulus of volume compression referred to twice the yield point. Moreover, the condition of coaxiality of the stress and strain tensors

$$\frac{\gamma_{rz}}{\varepsilon_r - \varepsilon_z} = \frac{2\tau_{rz}}{\sigma_r - \sigma_z} \tag{1.4}$$

and the condition of an elastic change in volume

$$\sigma_r + \sigma_z + \sigma_\varphi = 3k\,(\varepsilon_r + \varepsilon_z + \varepsilon_\varphi) \tag{1.5}$$

should be added to (1.1), (1.3).

The equilibrium equations (1.1), the plasticity condition (1.2) and the model relationships (1.3) – (1.5) generate a complete system of equations to determine the six functions σ_r, σ_z, σ_φ, τ_{rz}, u, v. Eliminating σ_φ from the equilibrium equations by using (1.3), as well as from the condition of the elastic change in volume (1.5) by using the Levi transformation

$$\sigma_r = \sigma + \frac{\cos 2\theta}{2}, \quad \sigma_z = \sigma - \frac{\cos 2\theta}{2}, \quad \tau_{rz} = \frac{\sin 2\theta}{2}, \quad \sigma = \frac{\sigma_1 + \sigma_2}{2} \tag{1.6}$$

where θ is the angle between the first principal direction and the r-axis, we obtain the following quasi-linear system of equations

$$\begin{gathered}
\frac{\partial \sigma}{\partial r} - \sin 2\theta \frac{\partial \theta}{\partial r} + \cos 2\theta \frac{\partial \theta}{\partial z} = -\frac{\cos 2\theta}{2r} - \frac{1-2\nu}{r}\sigma + 3k(1-2\nu)\frac{u}{r^2} \\
\frac{\partial \sigma}{\partial z} + \cos 2\theta \frac{\partial \theta}{\partial r} + \sin 2\theta \frac{\partial \theta}{\partial z} = -\frac{\sin 2\theta}{2r} \\
\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + 2\nu \frac{u}{r} = \frac{2(1+\nu)}{3k} \\
\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\cos 2\theta = \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)\sin 2\theta
\end{gathered} \tag{1.7}$$

The solution of the system (1.7) determines the functions σ, θ, u, v, after which σ_r, σ_z, τ_{rz} are determined from (1.6) and σ_φ from (1.3). It can be shown that the system (1.7) is of hyperbolic type with double characteristics whose equations are

$$\begin{aligned}
\frac{dr}{dz} &= \operatorname{ctg}\left(\theta - \frac{\pi}{4}\right) = \tau_1 = \tau_3 \quad (\alpha\text{- line}) \\
\frac{dr}{dz} &= \operatorname{ctg}\left(\theta + \frac{\pi}{4}\right) = \tau_2 = \tau_4 \quad (\beta\text{-line})
\end{aligned} \tag{1.8}$$

The characteristics are therefore surfaces of maximal shear stress (slip surfaces). It is seen that the system (1.7) can be separated: the first two equations contain derivatives of the functions σ, θ and the second two equations, derivatives of u, v, whence the duplicity of the characterisitcs (1.8) follows. The iteration process considered below will be constructed according to this separation.

Let us introduce differential operators along the characteristics

$$\begin{aligned}
\frac{d}{dz} &= \frac{\partial}{\partial z} + \operatorname{ctg}\left(\theta - \frac{\pi}{4}\right)\frac{\partial}{\partial r} \quad (\text{along } \alpha\text{-line}) \\
\frac{d}{dz} &= \frac{\partial}{\partial z} + \operatorname{ctg}\left(\theta + \frac{\pi}{4}\right)\frac{\partial}{\partial r} \quad (\text{along } \beta\text{-line})
\end{aligned} \tag{1.9}$$

and let us perform a number of manipulations with the equations of the system (1.7). Then we arrive at relationships on the characteristics

$$d[\sigma-\theta]=\frac{1}{r}\left[\frac{dz}{2}-(1-2\nu)\left(\sigma-3k\frac{u}{r}\right)dr\right] \tag{1.10}$$

$$d[\sigma+\theta]=-\frac{1}{r}\left[\frac{dz}{2}+(1-2\nu)\left(\sigma-3k\frac{u}{r}\right)dr\right]$$

$$dU-V\,d\theta=\left[\frac{1+\nu}{3k}\sigma-\nu\frac{u}{r}\right]ds_\alpha \tag{1.11}$$

$$dV+U\,d\theta=\left[\frac{1+\nu}{3k}\sigma-\nu\frac{u}{r}\right]ds_\beta$$

Here ds_α, ds_β are length elements along the α and β lines, U, V are the displacements along these lines, respectively. Let us note that (1.10) pass into Hencky relationships, and (1.11) into the Geiringer relationships of the plane problem of ideal plasticity theory as $r \to \infty$, $k \to \infty$ (incompressibility of ideally plastic material). It also follows from the relationships on the characteristics (1.10) that the first two equations of the system (1.7) are reducible to Riemann invariants. In fact, using the notation

$$\mathbf{t}=(t_1,\ t_2),\qquad t_1=\sigma-\theta,\qquad t_2=\sigma+\theta \tag{1.12}$$

we obtain

$$\frac{\partial t_k}{\partial z}+\tau_k(\mathbf{t})\frac{\partial t_k}{\partial r}=f_k(r,\mathbf{t})+g_k(r,\mathbf{t})u,\qquad k=1,2 \tag{1.13}$$

Let us rewrite (1.11) as

$$\mathbf{l}_k\left[\frac{\partial \mathbf{U}}{\partial z}+\tau_k(\mathbf{t})\frac{\partial \mathbf{U}}{\partial r}\right]=f_k(r,\mathbf{t})+g_k(r,\mathbf{t})u,\qquad k=3,4 \tag{1.14}$$

$$\mathbf{U}=(u,\ v),\qquad \mathbf{l}_3=(\tau_3,\ 1),\ \mathbf{l}_4=(\tau_4,\ 1)$$

$$f_1(r,\mathbf{t})=\frac{1}{2r}-\frac{1-2\nu}{2r}(t_1+t_2)\tau_1(\mathbf{t}) \tag{1.15}$$

$$g_1(r,\mathbf{t})=\frac{3k(1-2\nu)}{r^2}\tau_1(\mathbf{t}),\qquad \tau_1(\mathbf{t})=\operatorname{ctg}\left(\frac{t_2-t_1}{2}-\frac{\pi}{4}\right)$$

$$f_2(r,\mathbf{t})=-\frac{1}{2r}-\frac{1-2\nu}{2r}(t_1+t_2)\tau_2(\mathbf{t})$$

$$g_2(r,\mathbf{t})=\frac{3k(1-2\nu)}{r^2}\tau_2(\mathbf{t}),\qquad \tau_2(\mathbf{t})=\operatorname{ctg}\left(\frac{t_2-t_1}{2}+\frac{\pi}{4}\right)$$

$$f_3(r,\mathbf{t})=-\frac{1+\nu}{3k}(t_1+t_2)\frac{\tau_3(\mathbf{t})}{\cos(t_2-t_1)}$$

$$g_3(r,\mathbf{t})=\frac{2\nu}{r}\frac{\tau_3(\mathbf{t})}{\cos(t_2-t_1)},\qquad \tau_3(\mathbf{t})=\tau_1(\mathbf{t})$$

$$f_4(r,\mathbf{t})=\frac{1+\nu}{3k}(t_1+t_2)\frac{\tau_4(\mathbf{t})}{\cos(t_2-t_1)}$$

$$g_4(r,\mathbf{t})=-\frac{2\nu}{r}\frac{\tau_4(\mathbf{t})}{\cos(t_2-t_1)},\qquad \tau_4(\mathbf{t})=\tau_2(\mathbf{t})$$

Here and henceforth, the summation is only over the Greek subscripts (it is missing over the subscript k).

2. Approximate method of solving the Cauchy problem for the system (1.7) and its convergence. For simplicity, let us consider that the initial conditions of the Cauchy problem for the system (1.7) are given on a segment of the $z = 0$ axis, $a \leqslant r \leqslant b$, since the general Cauchy problem reduces to the problem under consideration by interchanging the independent r, z variables, which does not alter the form of the equations in (1.7).

Let the initial conditions

$$\sigma(r, 0) = \sigma_0(r), \qquad \theta(r, 0) = \theta_0(r)$$
$$u(r, 0) = u_0(r), \qquad v(r, 0) = v_0(r)$$

be given for the system (1.7) on the segment $[a, b]$ of the $z = 0$ axis. According to the invariants (1.12) introduced and the vector $\mathbf{U}$, this corresponds to the following initial conditions for the characteristic system (1.13), (1.14):

$$\mathbf{t}_0(r, 0) = (t_1^\circ(r), t_2^\circ(r)), \quad \mathbf{U}_0(r, 0) = (u_0(r), v_0(r)) \tag{2.1}$$

Existence and uniqueness theorems of the Cauchy problem for general quasi-linear systems of equations with two independent variables in the class C_1 have been considered in [3, 4]. We shall also assume here that $\mathbf{t}_0(r)$, $\mathbf{U}_0(r) \in C_1[a, b]$. The smoothness of the remaining input data f_k, g_k follows from their form (1.15) in the case under consideration for $r > 0$. We define the following iteration process to construct the solution of the problem (1.13), (1.14), (2.1). Let $\mathbf{U}^{(0)} = (u^{(0)}, v^{(0)})$ be an arbitrary vector function belonging to C_1 and such that

$$\mathbf{U}^{(0)}(r, 0) = \mathbf{U}_0(r) \tag{2.2}$$

Substituting $u^{(0)}(r, z)$ into the right side of the system (1.13), we determine $t^{(0)}$ as a solution of the Cauchy problem with the initial conditions

$$\mathbf{t}^{(0)}(r, 0) = \mathbf{t}_0(r) \tag{2.3}$$

after which the solution $t^{(0)}(r, z)$ found is substituted into (1.14) and the Cauchy problem

$$\mathbf{U}^{(1)}(r, 0) = \mathbf{U}_0(r)$$

is solved for a linear system. Then, the approximation $\mathbf{U}^{(1)}$ determined is substituted into the left side of (1.13) and $\mathbf{t}^{(1)}(r, z)$ is determined

$$\mathbf{t}^{(1)}(r, 0) = \mathbf{t}_0(r)$$

and this process is thus repeated many times. Let the approximation $\mathbf{U}^{(i)} \in C_1$ be constructed, then for $\mathbf{t}^{(i)}$ we have

$$\frac{\partial t_k^{(i)}}{\partial z} + \tau_k^{(i)} \frac{\partial t_k^{(i)}}{\partial r} = f_k^{(i)} + g_k^{(i)} u^{(i)}, \quad k = 1, 2 \tag{2.4}$$

$$\mathbf{t}^{(i)}(r, 0) = \mathbf{t}_0(r) \tag{2.5}$$

and we determine $\mathbf{U}^{(i+1)}$ from the solution of the Cauchy problem for the linear system

$$\mathbf{l}_k^{(i)} \left[\frac{\partial \mathbf{U}^{(i+1)}}{\partial z} + \tau_k^{(i)} \frac{\partial \mathbf{U}^{(i+1)}}{\partial r}\right] = f_k^{(i)} + g_k^{(i)} u^{(i+1)}, \quad k = 3, 4 \tag{2.6}$$

$$\mathbf{U}^{(i+1)}(r, 0) = \mathbf{U}_0(r) \tag{2.7}$$

It follows from the existence theorem for the solution of quasi-linear and linear systems [3, 4] that a solution $\mathbf{t}^{(i)}$, $\mathbf{U}^{(i+1)} \in C_1$ exists in the domain of definiteness $G^{(i)}$

of the Cauchy problems (2.4) – (2.7) so that all the approximations are defined and continuously differentiable in the domains $G^{(i)}$. We note that the domains of definiteness of the Cauchy problems (2.4) – (2.7) coincide since the characteristics of the system (2.4) are also characteristics of the system (2.6). Moreover, since the initial problem (1.13), (1.14), (2.1) is quasi-linear, the domain of definiteness G of the solution of this problem is determined simultaneously with the solution $\mathbf{U}(r, z)$, $\mathbf{t}(r, z)$ and is generally unknown in advance. However, according to [3] a domain $G \subseteq G$ of the variables (r, z) can be indicated in which the solution and its derivatives are known to remain bounded.

The first stage in the proof will consist of proving the existence of some domain G_0 belonging to all the domains $G^{(i)}$ and such that all the approximations and their first derivatives are bounded in this domain. To do this let us write down a continued system for the quasi-linear system (2.4) and a continued system for the linear system (2.6) (we omit this latter because of the total analogy in the computations). Let us recall that the continued system of hyperbolic quasi-linear equations is determined by differentiating the initial system with respect to the independent variables and result in Riemann invariants which are determined as follows in this case:

$$H_k^{(i)} = l_k{}^\alpha \frac{\partial n_\alpha^{(i)}}{\partial r} \qquad (\alpha,\ k = 1,\ 2)$$

$$\mathbf{l}_1 = (^1/_2,\ -^1/_2), \quad \mathbf{l}_2 = (1,\ 1), \quad \mathbf{n} = (n_1,\ n_2) = (\sigma,\ \theta)$$

The continued system for (2.4) then becomes

$$\frac{\partial H_k^{(i)}}{\partial z} + \tau_k^{(i)} \frac{\partial H_k^{(i)}}{\partial r} = L^k + L_\alpha^k H_\alpha^{(i)} + L_{\alpha\beta}^k H_\alpha^{(i)} H_\beta^{(i)} \tag{2.8}$$

$$\frac{\partial n_k^{(i)}}{\partial z} = F^k + F_\alpha^k H_\alpha^{(i)} \qquad (k,\ \alpha,\ \beta = 1,\ 2)$$

Here

$$L^k = \frac{(-1)^k}{r^2} + \frac{1-2\nu}{r^2}\, n_1^{(i)} \tau_k^{(i)} + 3k(1-2\nu) \frac{\partial}{\partial r}\left(\frac{u^{(i)}}{r^2}\right) \tag{2.9}$$

$$L_\alpha{}^k = \frac{1-2\nu}{2r}\left[-\tau_k^{(i)} + (-1)^{\alpha+1} \frac{\partial \tau_k^{(i)}}{\partial n_2} + (-1)^\alpha \frac{u^{(i)}}{r} \frac{\partial \tau_k^{(i)}}{\partial n_2}\right]$$

$$L_{\alpha\beta}^k = \frac{(-1)^{\alpha+\beta+k+1}}{2} \frac{\partial \tau_k^{(i)}}{\partial n_2}, \quad L_{22}{}^1 = L_{11}{}^2 = 0, \quad \alpha \geqslant \beta$$

$$F^1 = -\frac{1-2\nu}{2r}\left(\tau_1^{(i)} + \tau_2^{(i)}\right)\left(n_1^{(i)} - \frac{3k}{r} u^{(i)}\right)$$

$$F^2 = -\frac{1}{2r} - \frac{1-2\nu}{2r}\left(\tau_2^{(i)} - \tau_1^{(i)}\right)\left(n_1^{(i)} - \frac{3k}{r} u^{(i)}\right)$$

$$F_\alpha{}^1 = -\frac{1}{2}\tau_\alpha^{(i)}, \qquad F_\alpha{}^2 = \frac{(-1)^{\alpha+1}}{2}\tau_\alpha^{(i)}$$

In addition to the system (2.8), let us consider the system of ordinary differential equations

$$dP/dz = L_0(N) + L_1(N)P + L_2(N)P^2 \tag{2.10}$$

$$dN/dz = F_0(N) + F_1(N)P$$

in which the following notation has been introduced

$$L_0(N) = \max_{G_0(N),\, \|u\|\leqslant W} \|L\|, \qquad L = (L^1, L^2) \tag{2.11}$$
$$L_1(N) = \max_{G_0(N),\, \|u\|\leqslant W} \|L_\alpha{}^k\|, \quad L_2(N) = \max_{G_0(N)} \max_{\beta=1,2} \|L^k_{\alpha\beta}\|$$
$$F_0(N) = \max_{G_0(N),\, \|u\|\leqslant W} \|F\|, \quad F = (F^1, F^2), \quad F_1(N) = \max_{G_0(N)} \|F_\alpha{}^k\|$$
$$G_0(N) = \{a \leqslant r \leqslant b,\ 0 \leqslant z \leqslant z_0;\ \|\mathbf{n}\| \leqslant N\}$$

where $W = W(z)$ is determined from the solution of ordinary differential equations analogous to (2.10), constructed for the linear system (2.6). Following [3], we call the system (2.10) the majorant system. Let N_0, P_0 denote the quantities

$$N_0 = \max_{a\leqslant r\leqslant b} \|\mathbf{n}^\circ(r)\|, \qquad P_0 = \max_{a\leqslant r\leqslant b} \left\| \mathbf{l}_k \frac{\partial \mathbf{n}^\circ}{\partial r} \right\|$$

We give the initial conditions

$$P(0) = P_0, \qquad N(0) = N_0 \tag{2.12}$$

for the system (2.10). It follows from a comparison between (2.8) and (2.10) that if

$$\|\mathbf{n}^{(i)}\| \leqslant N(z), \quad \|\mathbf{H}^{(i)}\| \leqslant P(z), \quad \|u\| \leqslant W(z) \tag{2.13}$$

then

$$\left\| \frac{\partial \mathbf{n}^{(i)}}{\partial z} \right\| \leqslant \frac{dN}{dz}, \qquad \left\| \frac{\partial H_k^{(i)}}{\partial z} + \tau_k^{(i)} \frac{\partial H_k^{(i)}}{\partial r} \right\| \leqslant \frac{dP}{\partial z}$$

Since compliance with (2.13) for $z = 0$ follows from (2.12), then for any $z > 0$

$$\|\mathbf{n}^{(i)}\| \leqslant N(z), \qquad \|\mathbf{H}^{(i)}\| \leqslant P(z)$$

Thus, the functions $N(z)$, $P(z)$ majorize the growth of the approximation $\mathbf{n}^{(i)}(r, z)$ and its first derivatives. For $0 \leqslant z \leqslant z_0$ let the solution $N(z)$, $P(z)$ of the majorant system satisfying (2.12) remain bounded. Then for $0 \leqslant z \leqslant z_0$ the approximation $\mathbf{n}^{(i)}(r, z)$ and its derivatives are known to remain bounded.

Now, we assume that all the approximations $\mathbf{n}^{(k)}(r, z)$, $k = 1, 2, \ldots i$ satisfy the inequalities

$$\|\mathbf{n}^{(k)}\| \leqslant N(z), \quad \|\mathbf{H}^{(k)}\| \leqslant P(z), \quad \|\mathbf{U}^{(k)}\| \leqslant W(z) \tag{2.14}$$

and we show that (2.14) holds even for the $(i+1)$-th approximation. For this we write the system (2.8) for the $(i+1)$-th approximation

$$\frac{\partial H_k^{(i+1)}}{\partial z} + \tau_k^{(i+1)} \frac{\partial H_k^{(i+1)}}{\partial r} = L^k + L_\alpha{}^k H_\alpha^{(i+1)} + L^k_{\alpha\beta} H_\alpha^{(i+1)} H_\beta^{(i+1)} \tag{2.15}$$
$$\frac{\partial n_k^{(i+1)}}{\partial z} = F^k + F_\alpha{}^k H_\alpha^{(i+1)}$$

Let us hence note that the right sides of this system differ from the right sides of (2.8) by $u^{(i+1)}$ having been substituted in place of $u^{(i)}$. But since the coefficients of the majorant system (2.10) have been determined for u such that $\|u\| \leqslant W(z)$, where $W(z)$ is obtained from the solution of the majorant system for Eqs. (2.6), i.e. $\|u^{(i+1)}\| \leqslant W(z)$. Consequently this permits the conclusion that the system (2.10) is also a majorant system for (2.15). Hence, there follows

$$\|\mathbf{n}^{(i+1)}\| \leqslant N(z), \qquad \|\mathbf{H}^{(i+1)}\| \leqslant P(z) \tag{2.16}$$

Since the initial approximation can be selected so that (2.14) would be satisfied, then it has thereby been proved that all the approximations $\mathbf{n}^{(i+1)}$ satisfy the inequalities (2.16). It is shown analogously that all the approximations $\mathbf{U}^{(i+1)}$ and their derivatives are uniformly bounded. The existence of some domain G_0 belonging to all $G^{(i)}$ in which (2.16) is satisfied is thereby proved. Hence, if the relationships (2.11) are considered, and computations analogous to those considered in [3] are performed, then we obtain that $G_0 \subseteq G_1$.

The second stage of the proof consists of proving the uniform convergence of the sequence $\{\mathbf{t}^{(i)}\}$ in G_0. But let us first formulate the following lemma (see [3] for the proof).

Let a vector function $\mathbf{n}(z) = (n_1, n_2)$ continuous in the segment $0 \leqslant z \leqslant z_0$ satisfy the inequality

$$\|\mathbf{n}(z)\| \leqslant \int_0^z [A(\tau) + B(\tau) \max_{0\leqslant\xi\leqslant\tau} \|\mathbf{n}(\xi)\|]\, d\tau \tag{2.17}$$

and let $|A(z)| \leqslant A$, $|B(z)| \leqslant B$ for $0 \leqslant z \leqslant z_0$. Then the estimate

$$\|\mathbf{n}(z)\| \leqslant \max_{0\leqslant\tau\leqslant z} \|\mathbf{n}(\tau)\| \leqslant \frac{A}{B}(e^{Bz} - 1) \tag{2.18}$$

holds for $0 \leqslant z \leqslant z_0$.

It can be shown that the operator corresponding to (1.14) and transforming the approximation $\mathbf{t}^{(i)}$ into $\mathbf{U}^{(i+1)}$ is "compressing". To do this, we write (1.14) for the $\mathbf{U}^{(i)}$-approximation

$$\mathbf{l}_k^{(i-1)}\left[\frac{\partial \mathbf{U}^{(i)}}{\partial z} + \tau_k^{(i-1)} \frac{\partial \mathbf{U}^{(i)}}{\partial r}\right] = f_k^{(i-1)} + g_k^{(i-1)} u^{(i)}, \quad k = 3, 4 \tag{2.19}$$

and we introduce the expression

$$\rho_k^{(i+1)} = \mathbf{l}_k^{(i)}(\mathbf{U}^{(i+1)} - \mathbf{U}^{(i)}), \qquad \mathbf{U} = (U_1, U_2) = (u, v), \ \mathbf{l}_k = (l_k^3, l_k^4) \tag{2.20}$$

Making use of the theorem on finite increments, we have for the difference, for example:

$$l_k^{\ \alpha}\tau_k^{(i)} - l_k^{\ \alpha}\tau_k^{(i-1)} = \int_0^1 \frac{\partial(l_k^{\ \alpha}\tau_k)}{\partial t_\beta}(\mathbf{t}^{(i-1)} + \lambda(\mathbf{t}^{(i)} - \mathbf{t}^{(i-1)}))\, d\lambda\, (t_\beta^{(i)} - t_\beta^{(i-1)}) \tag{2.21}$$

Afterwards, we subtract (2.19) from (2.6) and we take into account the possibility of solving (2.20) for $U_k^{(i+1)} - U_k^{(i)}$ and relations of the form (2.21) for $l_k^{(i)} - l_k^{(i-1)}$; $f_k^{(i)} - f_k^{(i-1)}$; $g_k^i - g_k^{(i-1)}$. We thence obtain a linear system for $\rho_k^{(i-1)}$

$$\frac{\partial \rho_k^{(i+1)}}{\partial z} + \tau_k^{(i)} \frac{\partial \rho_k^{(i+1)}}{\partial r} = X_k^{\ \alpha}\rho_\alpha^{(i+1)} + \Lambda_k^{\ \beta}(t_\beta^{(i)} - t_\beta^{(i-1)}), \qquad k, \alpha = 3, 4; \quad \beta = 1, 2 \tag{2.22}$$

Here X_k^{α}, Λ_k^{β} are determined in terms of the functions f_k, g_k, l_k, τ_k, U_k and their derivatives. Integrating (2.22) along the characteristics $dr : dz = \tau_k^{(i)}$ lying in G_0, we have

$$|\rho_k^{(i+1)}| \leqslant \int_0^z |\Lambda_k^{\ \beta}(t_\beta^{(i)} - t_\beta^{(i-1)}) + X_k^{\ \alpha}\rho_\alpha^{(i+1)}|\, d\tau$$

Hence

$$\|X_k^{\ \alpha}\| \leqslant B, \quad \|\Lambda_k^{\ \beta}\| \leqslant B, \quad B = \text{const} \tag{2.23}$$

holds everywhere in G_0 by virtue of (2.13). We use the notation

$$R_{i+1}(z) = \max_{\tau,\, r\in G_0,\, \tau\leqslant z} \|\rho^{(i+1)}\|$$

Then it follows from (2.22), (2.23) that

$$R_{i+1}(z) \leqslant B\int_0^z [\|\mathbf{t}^{(i)} - \mathbf{t}^{(i-1)}\| + R_{i+1}(\tau)]\, d\tau$$

and we obtain by virtue of the lemma presented

$$R_{i+1}(z) \leqslant (e^{Bz} - 1)\|\mathbf{t}^{(i)} - \mathbf{t}^{(i-1)}\| \qquad (2.24)$$

Therefore, for small $0 \leqslant z \leqslant z_0$ the operator corresponding to the system (1.14) is "compressive". Using the estimate (2.24), we can show the uniform convergence of the sequence of approximations $\{\mathbf{t}^{(i)}\}$. In fact, let us write the system (1.13) for the $(i-1)$-th approximation

$$\frac{\partial t_k^{(i-1)}}{\partial z} + \tau_k^{(i-1)}\frac{\partial t_k^{(i-1)}}{\partial r} = f_k^{(i-1)} + g_k^{(i-1)}u^{(i-1)}, \quad k = 1, 2 \qquad (2.25)$$

Now if (2.25) is subtracted from (2.4) correspondingly, and relationships analogous to (2.21), obtained by using the theorem on finite increments, are used here, then to determine $\delta_k^{(i)} = t_k^{(i)} - t_k^{(i-1)}$ we obtain the linear system

$$\frac{\partial \delta_k^{(i)}}{\partial z} + \tau_k^{(i)}\frac{\partial \delta_k^{(i)}}{\partial r} = \Pi_k{}^\beta \delta_\beta^{(i)} + M_k{}^\alpha \rho_\alpha^{(i)}, \quad k, \beta = 1, 2;\ \alpha = 1, 2 \qquad (2.26)$$

Here, exactly as in (2.22), the $\Pi_k{}^\beta$, $M_k{}^\alpha$ are determined in terms of the functions f_k, g_k, $\mathbf{l}_k$, $\mathbf{t}$, $\mathbf{U}$ and their derivatives. Integrating (2.26) along the characteristics lying in G_0, we obtain for each point of G_0

$$|\delta_k^{(i)}| \leqslant \int_0^z |\Pi_k{}^\beta \delta_\beta^{(i)} + M_k{}^\alpha \rho_\alpha^{(i)}|\, d\tau$$

where the inequalities

$$\|\Pi_k{}^\beta\| \leqslant A, \quad \|M_k{}^\alpha\| \leqslant A, \quad A = \text{const}$$

hold for $\Pi_k{}^\beta$, $M_k{}^\alpha$ by virtue of (2.13). Then if we introduce

$$D_i(z) = \max_{\tau,\, r\in G_0,\, \tau\leqslant z} \|\delta^{(i)}\|$$

and use the inequality (2.24) written for $R_i(z)$, we obtain

$$D_i(z) \leqslant A\int_0^z [D_{i-1}(\tau) + D_i(\tau)]\, d\tau, \qquad D_i(z) \leqslant c\int_0^z D_{i-1}(\tau)\, d\tau$$

or

$$D_i(z) \leqslant \text{const}\,\frac{(cz)^{i-1}}{(i-1)!} \qquad (2.27)$$

which proves the uniform convergence of the sequence of approximations $\{\mathbf{t}^{(i)}\}$ in G_0. Let us note that the uniform convergence of the sequence $\{\mathbf{U}^{(i)}\}$ in G_0 follows from (2.24) and from (2.27).

Finally, using the continuous dependence of the solution of the Cauchy problem on the initial data, we obtain the uniform convergence of the sequence $\{\mathbf{H}^{(i)}\}$ and consequently of $\{\partial\mathbf{t}^{(i)}/\partial r\}$, $\{\partial\mathbf{t}^{(i)}/\partial z\}$, from an analysis of the continued system (2.8).

According to a known theorem of analysis, this means that the functions $\mathbf{t} = \lim \mathbf{t}^{(i)}$, $\mathbf{U} = \lim \mathbf{U}^{(i)}$, $i \to \infty$ are continuously differentiable in G_0. Passing to the limit in

(2.4), (2.6), we conclude that t, U is a solution of the problem (1.13), (1.14), (2.1).

Let us note the following:

1) The proof of the convergence of the proposed method of solving the Cauchy problem is carried over to the case of the characteristic problem as well as the mixed problem for the system (1.7) without essential change since it has been carried out by the method of characteristics.

2) The method considered for the solution of the incomplete plasticity equations can be applied to arbitrary hyperbolic quasi-linear systems with two independent variables admitting of separation in the above-mentioned sense.

3) The approximate method presented for solving the incomplete plasticity equations corresponding to the faces of the Tresca prism for the axisymmetric case reduces essentially to solving a number of plane problems of ideal plasticity theory (plane strain), whose numerical solution methods are quite well developed; the difference from the plane problem will consist only in the presence of an inhomogeneity in the equations under consideration (see (1.10), (1.11)).

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ON PLASTIC INSTABILITY IN SOME CASES OF SIMPLE FLOW

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The stability of deformation of an elastic viscoplastic hardening material under high precritical strains is investigated in a three-dimensional formulation. A solution of the stability equations is obtained in a rectangular coordinate system for a developed fundamental plastic flow process with small elastic strains in the case of a homogeneous precritical state. The surface and internal instability phenomena are investigated.

The papers [1, 2] are devoted to an investigation of the stability of deformation of an elastic-plastic material with large precritical strains. The stability of deformation of bodies of viscoplastic and elastic-viscoplastic material under